# Accuracy, Convergence and Adaptivity of MRAs

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November 26, 2007

# Outline

Function Spaces and Norms

Approximation Errors

Adaptivity of MRAs

We start with the Hilbert sapce

$$H = L_2(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{C} \mid \int_{\mathbb{R}^n} |u(x)|^2 dx < \infty\}.$$

In $H$ we have the scalar product (inner product)

$$\langle u, v \rangle = \int_{\mathbb{R}^n} u(x)\overline{v(x)} dx \text{ and the norm } \|u\|_2 = \sqrt{\langle u, u \rangle}.$$

The problem is that differential operators are not continuous in $H$. We are looking for functions with bounded energy, i.e. functions $\varphi \in L_2(\mathbb{R}^n)$ with

$$\langle \varphi, \mathcal{H}\varphi \rangle < \infty$$

where $\mathcal{H} = \mathcal{T} + \mathcal{V}$ is the Hamilton operator. Let us introduce the norm $\|\cdot\|_{H^1}$ via

$$\|u\|_{H^1}^2 := \|u\|_{L_2}^2 + \|\nabla u\|_{L_2}^2 = \langle u, u \rangle + \langle \nabla u, \nabla u \rangle = \langle u, u \rangle - \langle \Delta u, u \rangle$$

and the Sobolev space

$$H^1(\mathbb{R}^n) := \{u \in L_2(\mathbb{R}^n) \mid \|u\|_{H^1} < \infty\}.$$

## Theorem (Kato)

*For a wide class of potentials $\mathcal{V}$ including the Coulomb potential there holds:*
$\varphi \in L_2(\mathbb{R}^n)$ *has bounded energy iff* $\varphi$ *has bounded kinetic energy :*

$$\|\varphi\|_{H^1} < \infty.$$

*The Hamiltonian* $\mathcal{H}$ *is a bounded (continuous) operator*

$$\mathcal{H} : H^1(\mathbb{R}^n) \to (H^1(\mathbb{R}^n))',$$

*where* $(H^1(\mathbb{R}^n))'$ *is the dual space of* $H^1(\mathbb{R}^n)$ *and the corresponding norm is given by*

$$\|v\|_{H^{1'}} = \sup_{u \in H^1 \setminus \{0\}} \frac{|\langle v, u \rangle|}{\|u\|_{H^1}}.$$

Using Fourier transform you can see that

$$H^1(\mathbb{R}^n) = \{u \in L_2(\mathbb{R}^n) \mid \|(1 + |\cdot|^2)^{1/2}\hat{u}\|_{L_2(\mathbb{R}^n)} < \infty\}.$$

We generalize for $s \in \mathbb{R}$ the Sobolev space of order $s$

$$H^s(\mathbb{R}^n) := \{u \in L_2(\mathbb{R}^n) \mid \|u\|_{H^s} = \|(1 + |\cdot|^2)^{s/2}\hat{u}\|_{L_2(\mathbb{R})} < \infty\}$$

with the corresponding norm $\|u\|_{H^s}$. There holds the duality relation

$$(H^s(\mathbb{R}^n))' = H^{-s}(\mathbb{R}^n).$$

**Remark:** $H^0(\mathbb{R}^n) = L_2(\mathbb{R}^n)$

Function Spaces and Norms

Approximation Errors

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Let $E_0$ be the smallest eigenvalue of $\mathcal{H}$ and $\varphi_0$ the corresponding normalised eigenfunction. With the Ritz variation we have

$$E_0 = \min_{\langle\varphi,\varphi\rangle=1} \langle \varphi, \mathcal{H}\varphi \rangle,$$
$$\varphi_0 = \arg \min_{\langle\varphi,\varphi\rangle=1} \langle \varphi, \mathcal{H}\varphi \rangle.$$

Let $V_N$ be a $N$-dimensional subspace of $H^1(\mathbb{R}^n)$. We make a Ritz-Galerkin approximation

$$E_{0,N} = \min_{\substack{\varphi_N \in V_N \\ \langle\varphi_N,\varphi_N\rangle=1}} \langle \varphi_N, \mathcal{H}\varphi_N \rangle$$
$$\varphi_{0,N} = \arg \min_{\substack{\varphi_N \in V_N \\ \langle\varphi_N,\varphi_N\rangle=1}} \langle \varphi_N, \mathcal{H}\varphi_N \rangle.$$

## Theorem

*Then we have the inequalities*

$$|E_0 - E_{0,N}| \leq c_1 (\inf_{\varphi_N \in V_N} \|\varphi_0 - \varphi_N\|_{H^1})^2 \text{ and}$$

$$\|\varphi_0 - \varphi_{0,N}\|_{H^1} \leq c_2 \inf_{\varphi_N \in V_N} \|\varphi_0 - \varphi_N\|_{H^1} \text{ (quasi optimal convergence)} .$$

*with some constants* $c_1, c_2$.

In the following we omit the constants by writing

$$|E_0 - E_{0,N}| \lesssim (\inf_{\varphi_N \in V_N} \|\varphi_0 - \varphi_N\|_{H^1})^2$$

for example.

# Inequalities

If $f \in C^p(\mathbb{R}^d)$ and $\psi$ has $p$ vanishing moments we get with Taylor expansion

$$|f_k^j| = |\langle f, \psi_k^j \rangle| \lesssim 2^{-pj} 2^{-dj/2} \max_{i \in \{1,\dots,d\}} \sup_{\xi \in \operatorname{supp} \psi_k^j} |\frac{\partial^p}{\partial x_i^p} f(\xi)|.$$

Further it holds

$$\|\psi_k^j\|_{H^1} \sim 2^j \text{ ( chain rule )}$$
$$\|\psi_k^j\|_{H^s} \sim 2^{js}, \quad s \geq 0$$
$$\|f_j\|_{L_2} \lesssim 2^j \, \|f\|_{H^1}.$$

# Norm Equivalence

## Theorem

*Let*

$$\gamma = \sup\{s \geq 0 \mid \varphi \in H^s\}.$$

*It holds the fundamental norm equivalence for orthonormal wavelets and for all* $|s| < \gamma$

$$\|f\|_{H^s}^2 \sim \sum_k |f_k|^2 + \sum_{j=0}^{\infty} 2^{2js} \sum_k |f_k^j|^2$$

*with*

$$f_k = \langle f, \varphi_k \rangle, \qquad f_k^j = \langle f, \psi_k^j \rangle.$$

From the norm equivalence it follows the approximation estimation (AE)

$$\inf_{\varphi \in V^j} \|f - \varphi\|_{H^s} \lesssim 2^{-j(t-s)} \|f\|_{H^t}, \quad -p \leq s \leq t \leq p$$

and the inverse estimation (IE)

$$\|f_j\|_{H^s} \lesssim 2^{-j(t-s)} \|f\|_{H^t}, \quad -\gamma < s \leq t < \gamma.$$

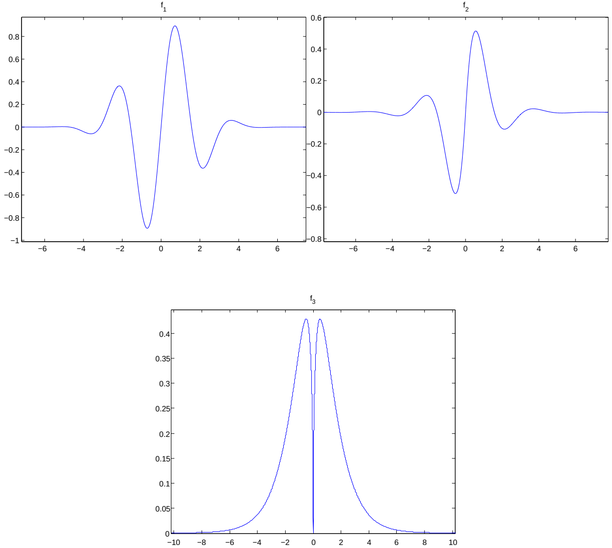
# Examples

Let us consider the following functions.

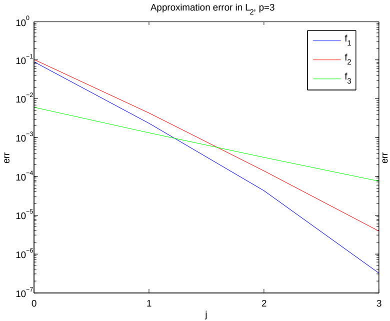
$$f_1(x) = \sin(2x)e^{-0.2x^2}$$
$$f_2(x) = \sin(2x)e^{-|x|}$$
$$f_3(x) = \sqrt{|x|}e^{-|x|}$$

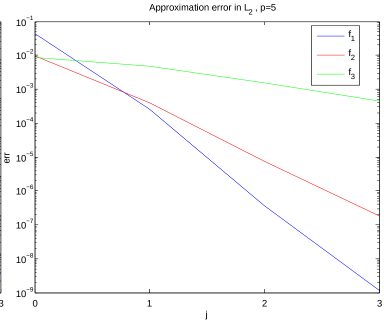
We have $f_1 \in H^s$ for all $s \geq 0$. $f_2 \in H^2 \setminus H^3$, $f_3 \notin H^1$.

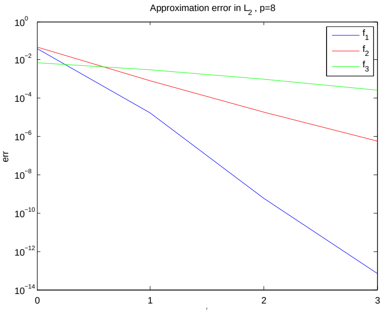
# Examples



# Approximation Errors



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Adaptivity of MRAs

Consider a Riesz (quasi-orthogonal) basis $\Psi = (\psi_\lambda)_{\lambda \in \mathcal{I}}$ in some Hilbert space $\mathrm{H}$.
For $v \in \mathrm{H}$, we consider the **error** of the **best n-term approximation**

$$\sigma_{n,\mathrm{H},\Psi}(v) = \min\{\|v - \sum_{\lambda \in \mathcal{T}} w_\lambda \psi_\lambda\| : w_\lambda \in \mathbb{C} \ , \ \mathcal{T} \subset \mathcal{I} \ , \ \sharp\mathcal{T} = n\}.$$

$$\mathrm{A}^s(\mathrm{H}, \Psi) := \{v \in \mathrm{H} : \sigma_{n,\mathrm{H},\Psi}(v) \lesssim n^{-s}\}$$

with

$$\|v\|_{\mathrm{A}^s} := \sup_{n \in \mathbb{N}} n^s \sigma_{n,\mathrm{H},\Psi}(v) + \|v\|_H$$

is a (quasi-)normed space.
**Approximation strategy**: Keep the $n$ largest coefficients $w_\lambda$ in $v = \sum_{\lambda \in \mathcal{I}} w_\lambda \psi_\lambda$.

With

$$h = 2^{-j}, \quad N \sim 2^{jd}$$

we have for sufficiently smooth $\varphi_0$

$$\|\varphi_0 - \varphi_N\|_{H^1} \lesssim h^{p-1} \sim N^{-\frac{p-1}{d}}$$
$$|E_0 - E_{0,N}| \lesssim h^{2(p-1)} \sim N^{-2\frac{p-1}{d}}.$$

# References


- S. Mallat, A Wavelet Tour of Signal Processing, 2nd. ed., Academic Press, 1999
- A. Cohen, Numerical Analysis of Wavelet Methods, Elsevier Science B.V., 2003