Daubechies Wavelets and Interpolating Scaling Functions and Application on PDEs

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Outline

Daubechies Wavelets

Conditions that fulfill the Daubechies wavelets:

1. They perform an orthonormal basis.
2. They have p vanishing moments for a given $p \in \mathbb{N}$.
3. They have the minimal support of all functions that fulfill the first two conditions.
4. They are rather smooth.

Fourier Space

The Fourier transform of a function f is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-i\omega x} dx$$

Given a scaling function φ and its corresponding wavelet ψ with its refinement equations

$$\varphi = \sum_k h_k \varphi_k^1$$
$$\psi = \sum_k g_k \varphi_k^1.$$

In Fourier space it gets to

$$\hat{\varphi}(\omega) = \frac{\sqrt{2}}{2} h(\frac{\omega}{2}) \hat{\varphi}(\frac{\omega}{2}) \tag{1}$$

$$\hat{\psi}(\omega) = \frac{\sqrt{2}}{2} g(\frac{\omega}{2}) \hat{\varphi}(\frac{\omega}{2})$$

with

$$h(\omega) = \sum_{k \in \mathbb{Z}} h_k e^{-ik\omega}$$

$$g(\omega) = \sum_{k \in \mathbb{Z}} g_k e^{-ik\omega}.$$

By succesive application of (**??**) we get

$$\hat{\varphi}(\omega) = \hat{\varphi}(0) \prod_{k=1}^{\infty} \frac{h(2^{-k}\omega)}{\sqrt{2}}$$

if $\hat{\varphi}$ is continuos at 0.

Computation of the Daubechies Wavelets

There is no analytical formula for the Daubechies scaling functions and wavelets. But you can compute exactly the filters h and g. They are tabulated in the internet, for example at wikipedia.

Computation of the Daubechies Wavelets

With the filters you can use the formulae in the Fourier space.

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} \frac{h(2^{-k}\omega)}{\sqrt{2}}$$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}} g(\frac{\omega}{2}) \hat{\varphi}(\frac{\omega}{2})$$

with

$$h(\omega) = \sum_{k \in \mathbb{Z}} h_k e^{-ik\omega}$$

$$g(\omega) = \sum_{k \in \mathbb{Z}} g_k e^{-ik\omega}.$$

φ

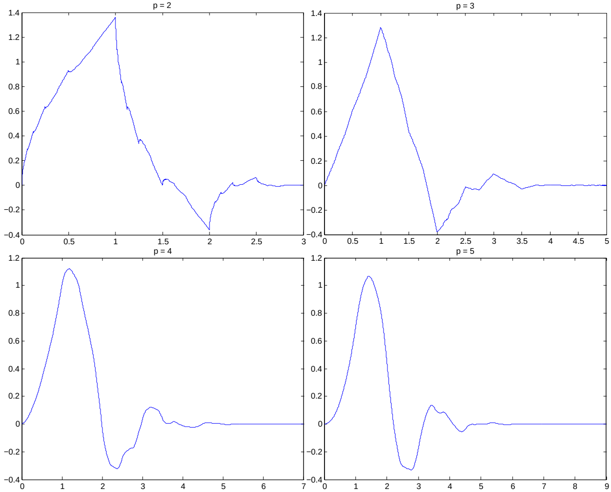


Figure: Daubechies scaling functions φ with $p = 2, 3, 4$ and 5.

ψ

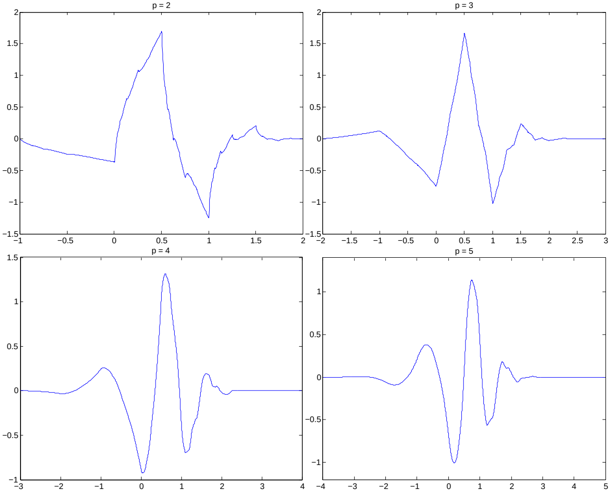


Figure: Daubechies wavelets ψ with $p = 2, 3, 4$ and 5.

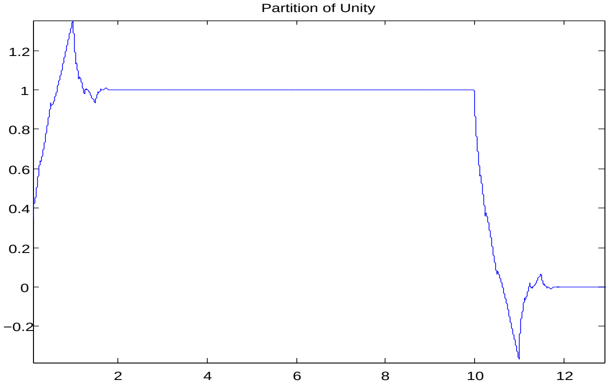
Vanishing Moments

A very important property of the Daubechies wavelets are the vanishing moments

$$\int_{\mathbb{R}} x^k \psi(x) dx = 0, \quad k = 0, \ldots, p-1.$$

That also means that every polynomial of degree at most $p-1$ is in V_0 in a compact subset K of $\mathbb{R}$. For example is $f = \mathbf{1}_K \in V_0\big|_K$. The series $(\sum_{k=-n}^{n} \varphi_k(x))_{n\in\mathbb{N}}$ is converging pointwise to the unity.

Partition of Unity



In the picture is shown the function

$$f = \sum_{k=0}^{10} \varphi_k$$

with φ the Daubechies scaling function of order $p = 2$.

An interpolating scaling function ϕ fulfills

$$\phi(k) = \delta_k, \quad k \in \mathbb{Z}.$$

Since the Daubechies scaling functions φ_k are orthonormal we get an interpolating scaling function ϕ (Interpolet) by folding the Daubechies function with itself.

$$\phi(t) = \int_{\mathbb{R}} \varphi(u)\varphi(u-t)du = \varphi \star \overline{\varphi}(t),$$

where

$$\overline{\varphi}(t) = \varphi(-t).$$

The interpolation property can be easily seen:

$$\phi(k) = \int_{\mathbb{R}} \varphi(u)\varphi(u-k)du = \langle \varphi, \varphi_k \rangle = \delta_k.$$

For the filter h^I corresponding to ϕ we get

$$h^I_k = (h \star \overline{h})_k.$$

To hold the interpolating property for $j \in \mathbb{Z}$ we omit the normalisation factor:

$$\phi^j_k(x) = \phi(2^j x - k).$$

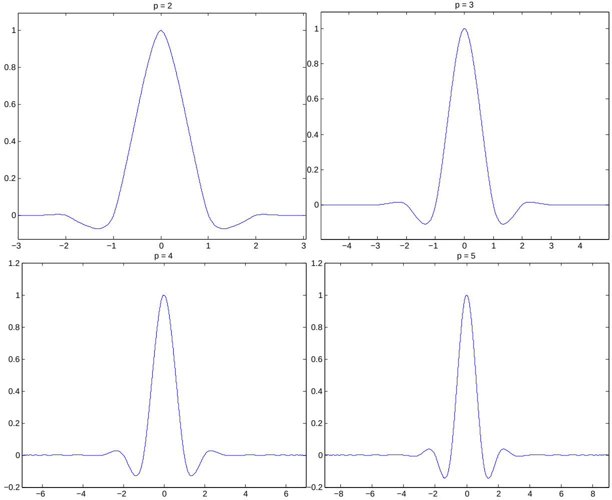


Figure: Interpolating scaling function ϕ for $p = 2, 3, 4$ and 5.

Interpolation Projection

With Interpolets we have a very simple projection P^J onto V^J:

$$P^J f = \sum_{k \in \mathbb{Z}} f(x_k) \phi_k^J, \quad x_k = 2^{-J} k$$

This projection is called interpolation projection or collocation projection.

Moments

We want to compute the moments

$$\mathcal{M}_k = \int_{\mathbb{R}} x^k \varphi(x) dx = \int_0^{2p-1} x^k \varphi(x) dx.$$

By using the refinement equation we get

$$\begin{aligned}
\mathcal{M}_k =& \sqrt{2} \sum_l h_l \int_{\mathbb{R}} x^k \varphi(2x-l) dx \\
=& \frac{\sqrt{2}}{2^{k+1}} \sum_l h_l \int_{\mathbb{R}} (x+l)^k \varphi(x) dx \\
=& \frac{\sqrt{2}}{2^{k+1}} \sum_l h_l \sum_{j=0}^{k} \binom{k}{j} l^j \mathcal{M}_{k-j}.
\end{aligned}$$

Using the fact that $\mathcal{M}_0 = 1$ we get the recursion formula

$$\mathcal{M}_k = \frac{1}{\sqrt{2}(2^k - 1)} \sum_{l=0}^{2p-1} h_l \sum_{j=1}^{k} \binom{k}{j} l^j \mathcal{M}_{k-j} \text{ for } k > 0.$$

With the $\mathcal{M}_k$ we can easily compute

$$\begin{aligned}
\langle x^k, \varphi_l^j \rangle &= \int_{\mathbb{R}} x^k \varphi_l^j(x) dx \\
&= 2^{-jk-j/2} \int_{\mathbb{R}} (x+l)^k \varphi(x) dx \\
&= 2^{-jk-j/2} \sum_{m=0}^{k} \binom{k}{m} l^m \mathcal{M}_{k-m}.
\end{aligned}$$

Therefore for any polynomial π the integral

$$\langle \pi, \varphi_l^j \rangle = \int_{\mathbb{R}} \pi(x)\varphi_l^j(x)dx$$

is exactly computable. For computing

$$\langle f, \varphi_l^j \rangle$$

where you know the function values of some nodes of f, you can do a polynomial interpolation with a polynomial π_f and integrate

$$\int_{\mathbb{R}} \pi_f(x)\varphi_l^j(x)dx \approx \int_{\mathbb{R}} f(x)\varphi_l^j(x)dx.$$

Computing $\langle D\varphi^j, D\varphi^j_k\rangle$

φ' satisfies a refinement equation, too:

$$D\varphi = \sum_k h_k D\varphi^1_k.$$

We can use this to compute

$$a_k = \langle D\varphi, D\varphi_k\rangle.$$

$$a_k = \sum_l \sum_m h_l h_m \langle D\varphi_l^1, D\varphi_{2k+m}^1 \rangle$$
$$= 4 \sum_l \sum_m h_l h_{m-2k} \langle D\varphi, D\varphi_{m-l} \rangle$$
$$= 4 \sum_l (h_l \sum_m h_{l+m-2k}) a_l$$

If you set $H_{k,l} = 4h_l \sum_m h_{l+m-2k}$ you have the eigen equation

$$Ha = a$$

i.e. you have to compute an eigenvector for the eigenvalue 1. It can be shown that this eigenvector is unique up to normalisation. To find the correct normalisation of the $a_k's$ we use the equation

$$\sum_k k^2 a_k = 2,$$

which holds for $p \geq 3$.

For $a_k^j = \langle D\varphi^j, D\varphi_k^j \rangle$ you have

$$a_k^j = 2^{2j} \langle D\varphi, D\varphi_k \rangle.$$

Remark: It holds

$$a_k^j = -\Delta\phi_k^j(0).$$

Computing $\langle \varphi, \phi_k \rangle$

With the same trick as before you can compute

$$\begin{aligned} b_k =& \langle \varphi, \phi_k \rangle \\ =& \sum_l \sum_m h_l h^I_m \langle \varphi^1_l, \phi^1_{2k+m} \rangle \\ =& 2^{-1/2} \sum_l \sum_m h_l h^I_{m-2k} \langle \varphi, \phi_{m-l} \rangle \\ =& 2^{-1/2} \sum_l (h_l \sum_m h^I_{l+m-2k}) b_l. \end{aligned}$$

With $\bar{H}_{k,l} = 2^{-1/2} \sum_m h^I_{l+m-2k}$ we have the equation

$$\bar{H} b = b,$$

i.e. we search an eigenvector of $\bar{H}$ for the eigenvalue 1. Here you also have to find a further information to achieve a unique solution. For $b^j_k = \langle \varphi^j, \phi^j_k \rangle$ we get

$$b^j_k = 2^{-j/2} b_k.$$

Until now we only considered one dimensional functions. If φ is a scaling function we define a d-dimensional scaling function Φ as follows.

$$\Phi(x_1, \ldots, x_d) = \prod_{i=1}^{d} \varphi(x_i)$$

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $j \in \mathbb{Z}$ we define

$$\Phi_{\mathbf{k}}^j(x_1, \ldots, x_d) = \prod_{i=1}^{d} \varphi_{k_i}^j(x_i).$$

Then with

$$\mathbf{V}^j = \overline{\operatorname{span}\{\Phi_{\mathbf{k}}^j \mid \mathbf{k} \in \mathbb{Z}^d\}} = \overline{\bigotimes_{i=1}^{d} V^j}$$

we get a MRA for $L_2(\mathbb{R}^d)$.

Wavelet Spaces

The wavelet spaces are a little bit more complicated. With $W_0^j = V^j$ and $W_1^j = W^j$ we have

$$\begin{aligned} \mathbf{V}^J &= \overline{\bigotimes_{i=1}^{d} V^J} \\ &= \overline{\bigotimes_{i=1}^{d} (V^{J-1} \oplus W^{J-1})} \\ &= \overline{\bigoplus_{\mathbf{e}\in\{0,1\}^d} \bigotimes_{i=1}^{d} W_{e_i}^{J-1}}. \end{aligned}$$

That means that in d dimensions we have $2^d - 1$ different wavelets. For example, for $d = 3$, we have

$$\begin{aligned}
\psi_1 =& \varphi \otimes \varphi \otimes \psi, \\
\psi_2 =& \varphi \otimes \psi \otimes \varphi, \\
\psi_3 =& \varphi \otimes \psi \otimes \psi, \\
\psi_4 =& \psi \otimes \varphi \otimes \varphi, \\
\psi_5 =& \psi \otimes \varphi \otimes \psi, \\
\psi_6 =& \psi \otimes \psi \otimes \varphi, \\
\psi_7 =& \psi \otimes \psi \otimes \psi.
\end{aligned}$$

Isotropic and Anisotropic Structures

If we pass to more levels, we can do it in two ways.

1. isotrop:

$$\mathbf{V}^J = \overline{\bigotimes_{i=1}^{d} V^0 \oplus \bigoplus_{j=0}^{J-1} \bigoplus_{\substack{\mathbf{e}\in\{0,1\}^d \\ \mathbf{e}\neq\mathbf{0}}} \bigotimes_{i=1}^{d} W_{e_i}^j}$$

2. anisotrop: With the notation $W^{-1} = V^0$

$$\mathbf{V}^J = \overline{\bigoplus_{j_1=-1}^{J-1} \bigoplus_{j_2=-1}^{J-1} \cdots \bigoplus_{j_d=-1}^{J-1} \bigotimes_{i=1}^{d} W^{j_i}}$$

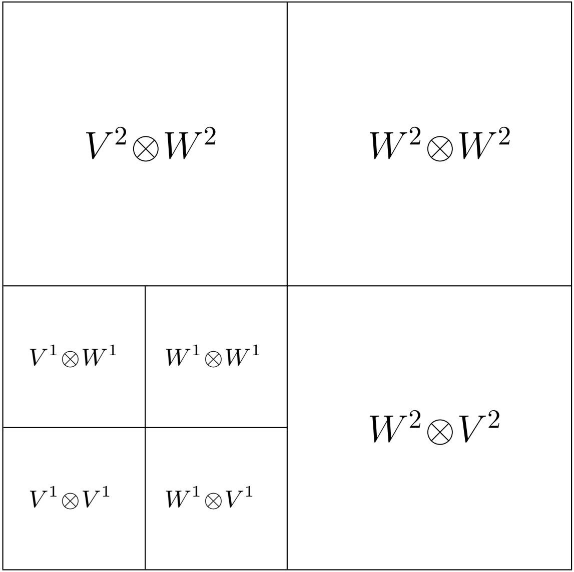
Isotropic Decomposition

$$V^3 \otimes V^3$$

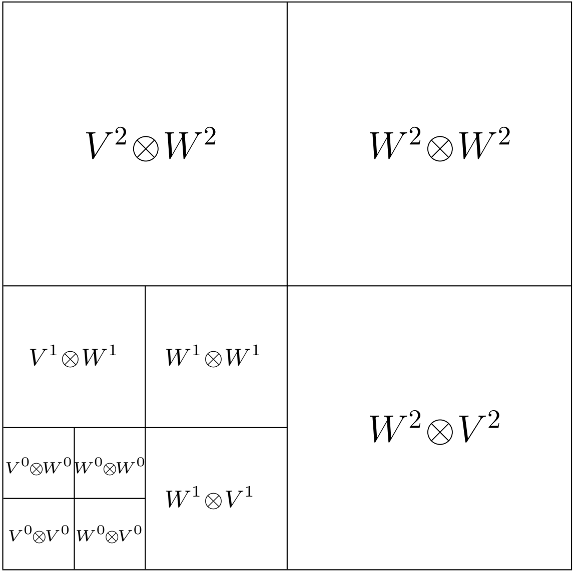
Isotropic Decomposition

$V^2 \otimes W^2$	$W^2 \otimes W^2$
$V^2 \otimes V^2$	$W^2 \otimes V^2$

Isotropic Decomposition



Isotropic Decomposition



Anisotropic Decomposition

$$V^3 \otimes V^3$$

Anisotropic Decomposition

$V^2 \otimes W^2$	$W^2 \otimes W^2$
$V^2 \otimes V^2$	$W^2 \otimes V^2$

Anisotropic Decomposition

$V^1 \otimes W^2$	$W^1 \otimes W^2$	$W^2 \otimes W^2$
$V^1 \otimes W^1$	$W^1 \otimes W^1$	$W^2 \otimes W^1$
$V^1 \otimes V^1$	$W^1 \otimes V^1$	$W^2 \otimes V^1$

Anisotropic Decomposition

$V^0 \otimes W^2$	$W^0 \otimes W^2$	$W^1 \otimes W^2$	$W^2 \otimes W^2$
$V^0 \otimes W^1$	$W^0 \otimes W^1$	$W^1 \otimes W^1$	$W^2 \otimes W^1$
$V^0 \otimes W^0$	$W^0 \otimes W^0$	$W^1 \otimes W^0$	$W^2 \otimes W^0$
$V^0 \otimes V^0$	$W^0 \otimes V^0$	$W^1 \otimes V^0$	$W^2 \otimes V^0$

Galerkin Scheme

Given an equation

$$Au - f = 0 \tag{2}$$

for a given function f and operator A. We are looking for the solution u in a Hilbert space H. (**??**) is equivalent to

$$\langle Au - f, v \rangle = 0 \quad \text{for all } v \in H.$$

Galerkin Scheme

We want to approximate the solution u in a N-dimensional subspace of H and denote the approximated solution by u_N. Let $\{v_1, \ldots, v_N\}$ be an orthonormal basis for V_N. We are getting the equations

$$\langle Au_N - f, v_i \rangle = 0, \quad i = \{1, \ldots, n\}.$$

We make the ansatz $u_N = \sum_{k=1}^{N} \lambda_k v_k$. We define the matrix $\mathbf{A}$ and the vector $\mathbf{f}$ by

$$A_{k,l} = \langle Av_k, v_l \rangle, \quad f_k = \langle f, v_k \rangle.$$

Then we have to solve the linear system

$$\mathbf{A}\lambda = \mathbf{f}.$$

Collocation Scheme

We make the ansatz

$$u_N = \sum_k u(x_k) v_k.$$

Here the v_k dont have to be orthonormal. Here are the Interpolets ϕ_k^j senseful. This leads to the equations

$$(Au_N - f)(x_k) = 0, \quad k = 1, \ldots, N.$$

By defining $\mathbf{A}$ and $\mathbf{f}$ by

$$A_{k,l} = (Av_l)(x_k), \quad f_k = f(x_k)$$

we get the linear system

$$\mathbf{Au} = \mathbf{f}, \text{ with } u_k \approx u(k).$$

References

- S. Mallat, A Wavelet Tour of Signal Processing, 2nd. ed., Academic Press, 1999
- A. Cohen, Numerical Analysis of Wavelet Methods, Elsevier Science B.V., 2003